| ISSN: 2395-7852 | www.ijarasem.com | Impact Factor: 5.649 |Bimonthly, Peer Reviewed & Referred Journal|

| Volume 7, Issue 2, March 2020 |

The History and Concept of Function in Mathematics

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ABSTRACT: Several fields of mathematics deal directly or indirectly with functions: mathematical analysis considers functions of one, two, or n variables, studying their properties as well as those of their derivatives; the theories of differential and integral equations aim at solving equations in which the unknowns are functions; functional analysis works with spaces made up of functions; and numerical analysis studies the processes of controlling the errors in the evaluation of all different kinds of functions. Other fields of mathematics deal with concepts that constitute generalizations or outgrowths of the notion of function; for example, algebra considers operations and relations, and mathematical logic studies recursive functions. It has long been argued that functions should constitute a fundamental concept in secondary school mathematics (Klein, 1908/1945) and the most recent curriculum orientations clearly emphasize the importance of functions (National Council of Teachers of Mathematics, 1989). Depending on the dominant mathematical viewpoint, the notion of function can be regarded in a number of different ways, each with different educational implications.

KEYWORDS: history,concept,mathematics,function,logic,operations

I. INTRODUCTION

The concept of function is rightly considered as one of the most important in all of mathematics. As the point, the line, and the plane were the basic elements of Euclidean geometry, the dominant theory from the time of Ancient Greece until the Modern Age, the notions of function and derivative constitute the foundation of mathematical analysis, the theory that has become central in the development of mathematics since then. Particular instances of functions may be found in ancient epochs; for example, counting, implying a correspondence between a set of given objects and a sequence of counting numbers; the four elementary arithmetical operations, which are functions of two variables; and the Babylonian tables of reciprocals, squares, square roots, cubics, and cubic roots. Historically, some mathematicians can be regarded as having foreseen and come close to a modern formulation of the concept of function. Among them is Oresme (1323-1382), who developed a geometric theory of latitudes of forms representing different degrees of intensity and extension. In his theory, some general ideas about independent and dependent variable, quantities seem to be present. But the emergence of functions in mathematics research as a clearly individualized concept and as an object of study in its own right is quite recent, dating to the end of the 17th century. The emergence of a notion of function as an individualized mathematical entity can be traced to the beginnings of infinitesimal calculus. Descartes (1596-1650) clearly stated that an equation in two variables, geometrically represented by a curve, indicates a dependence between variable quantities. The idea of derivative came about as a way of finding the tangent to any point of this curve. Newton (1642-1727) was one of the first mathematicians to show how functions could be developed in infinite power series, thus allowing for the intervention of infinite processes. He used "fluent" to designate independent variables, "relata quantitas" to indicate dependent variables, and "genita" to refer to quantities obtained from others using the four fundamental arithmetical operations. It was Leibniz (1646-1716) who first used the term "function" in 1673. He took function to designate, in very general terms, the dependence of geometrical quantities such as subtangents and subnormals on the shape of a curve. He also introduced the terms "constant," "variable," and "parameter." With the development of the study of curves by algebraic methods, a term to represent quantities that were dependent on one variable by means of an analytical expression was increasingly necessary. Finally, "function" was adopted for that purpose in the correspondence interchanged by Leibniz and Jean Bernoulli (1667-1748) between 1694 and 1698. The term function did not appear in a mathematics lexicon published in 1716. Two years later, however, Jean Bernoulli published an article, which would have widespread dissemination, containing his definition of a function of a variable as a quantity that is composed in some way from that variable and constants. Euler (1707-1793), a former student of Bernoulli, later added his touch to this definition speaking of analytical expression instead of quantity. The notion of function was therefore identified in practice with the notion of analytical expression. This formulation was soon perceived to lead to several incoherences; in fact, the same function could be represented by several different analytical expressions. The formulation also yielded serious limitations on the classes of functions that could be considered. In present day terminology, we can say that Euler's definition included just the analytic functions, a restricted subset of the already small class of continuous functions. Aware of these shortcomings, Euler proposed an alternative definition that did not attract much attention at the time. As far as mainstream mathematics is concerned, the identification of functions with analytical expressions would remain unchanged for all of the 18th century. In the 19th century, however, the notion of function underwent successive enlargements and clarifications that deeply changed its nature and meaning.[1,2,3]

Other mathematicians gave their own versions of the definition of a function. Condorcet seems to have been the first to take up Euler's general definition of 1755. In 1778 the first two parts of Condorcet intended five part work Trait´e du calcul integral was sent to the Paris Academy. It was never published but was seen by many leading French mathematicians. In this work Condorcet distinguished three types of functions: explicit functions, implicit functions given only by unsolved equations, and functions which are defined from physical considerations such as being the solution to a differential equation. Cauchy, in 1821, came up with a definition making the dependence between variables central to the function concept in Cours d'analyse. Despite the generality of Cauchy's definition, which was designed to cover the case of explicit and implicit functions, he was still thinking of a function in terms of a formula. In fact he makes the distinction between explicit and implicit functions immediately after giving this definition. He also introduces concepts which indicate that he is still thinking in terms of analytic expressions. Fourier, in Th´eorie analytique de la Chaleur in 1822, gave a definition which deliberately moved away from analytic expressions. However, despite this, when he begins to prove theorems about expressing an arbitrary function as a Fourier series, he uses the fact that his arbitrary function is continuous in the modern sense! Dirichlet, in 1837, accepted Fourier's definition of a function and immediately after giving this definition he defined a continuous function (using continuous in the modern sense). Dirichlet also gave an example of a function defined on the interval [0, 1] which is discontinuous at every point, namely f(x) = (0 if x is rational 1 if x is irrational Around this time many pathological functions were constructed. Cauchy gave an early example when he noted that f(x) = (e −1/x2 for x 6= 0, 0 x = 0, is a continuous function which has all its derivatives at 0 equal to 0. It therefore has a Taylor series which converges everywhere but only equals the function at 0.

II. DISCUSSION

Another important contribution to the evolution of function came from the works of Fourier (1768-1830), who was concerned with the problem of heat flow in material bodies. Fourier considered temperature as a function of two variables, namely time and space. At some point, he conjectured that it would be possible to obtain a development of any function in a trigonometric series in a suitable interval. Fourier, however, never gave a mathematical proof of his assertion. The problem was later taken up by Dirichlet (1805-1859) who formulated the sufficient conditions so that a function may be represented by a Fourier series. To do so, Dirichlet needed to separate the concept of function from its analytical representation. He did this in 1837, casting the definition of function in terms of an arbitrary correspondence between variables representing numerical sets. A function, then, became a correspondence between two variables so that to any value of the independent variable, there is associated one and only one value of the dependent variable. Dirichlet also gave this well-known example of a function that is discontinuous in all the points of the domain [0,1]: 0 if x is a rational number 1 otherwise. With the development of set theory, initiated by Cantor (1845-1918), the notion of function continued to evolve. [5,7,8]In the 20th century, function was extended to include all arbitrary correspondences satisfying the uniqueness condition between sets, numerical or nonnumerical. The evolution of function has continued. From the notion of correspondence, mathematicians moved to the notion of relation. A close relative to the notion of function constitutes a primitive concept in category theory. In the theory of computation, for example, as in l-calculus, a function is not viewed as a relation but as a computational rule. In its beginnings, the notion of function was used to designate correspondences between geometrical entities. Through its association with the study of analytical expressions, function secured a fundamental place in the mainstream of mathematical thinking. To indicate the historical role of this association, Youschkevitch (1976/77) commented, It was the analytical method of introducing functions that revolutionized mathematics and, because of its extraordinary efficiency, secured a central place for the notion of function in all exact sciences. (p. 39) This association between analytical expressions and geometrical objects, in fact, revealed itself to be so highly fruitful that it still informs current mathematical practice.

The notion of function did not appear in mathematics by chance. It arose, as Bento Caraça (1951) so masterfully has shown, as the necessary mathematical tool for the quantitative study of natural phenomena, begun by Galileo (1564-1642) and Kepler (1571-1630). Its development was based in the expressive possibilities enabled by the modern algebraic notation created by Viète (1540-1603) and, especially, by the analytic geometry introduced by Descartes and Fermat (1601-1665). In opposition to the verbalistic stance of the medieval scholastic thinking, Galileo underscored that mathematics was the most appropriate language for the study of nature. According to Galileo, to study a given phenomenon, it was necessary to measure quantities, identify regularities, and f(x) = { obtain relationships representing mathematical descriptions as simply as possible. The study of the movement of falling bodies, of the motion of planets, and more generally, of curvilinear motion, led to the consideration of direct and inverse proportionalities, as well as of polynomial and trigonometric functions. Mathematics and physics were at this point closely interconnected. Newton, regarded as one of the greatest mathematicians of all times, was also a prominent physicist. Many other mathematicians, such as Daniel Bernoulli, Euler, Lagrange, and Fourier were also very interested in physical problems. One of the most extraordinary discoveries of Newton provides an example on the nature of variation. Newton found that the law of variation of the motion of a material body of mass m, which in modern notation may be given by the function s(t), the space varying as a function of time, does not have a direct relationship with the force f acting upon that body. A simple relationship does not exist for the law of velocity given by v(t), in which v is the derivative of s, ds/dt. Such a relationship does exist, however, for the acceleration of the body, given by a(t), the second derivative of s, that is, dv/dt, and is expressed by a very simple law: f = ma. Functions are excellent tools to study problems of variation. A given quantity may vary in time, may vary in space, may vary with other quantities, and may even vary simultaneously in several dimensions. Such variation may be faster or slower, or may even disappear at some point. It may follow simple or complex patterns and obey very diverse restrictions. In its origin, the notion of function was associated with the notion of natural law. [9,10,11]The idea of regularity was one of its more important constituent elements. In fact, we may consider three essential elements in the formation of the primitive 17th and 18th century concept of function: (a) the algebraic notation, carrying important aspects such as simplicity and rigor, allowing the manipulation of analytical expressions and condensing in itself a large quantity of information; (b) the geometrical representation, yielding a fundamental intuitive basis, of which a remarkable example is the association of the notions of tangent to one curve and derivative of a function; and (c) the connection with the concrete problems of the physical world, associated with the idea of regularity, providing the fundamental motivation and interest for the study of families of functions. After some time, mathematicians began considering functions to which no analytical expression corresponded, functions that did not have a simple geometrical representation, or even functions that did not have any relation to concrete physical situations. Thus, the concept of function began evolving on its own, moving away from its origins. That evolution had its dramatic moments. We may remember the moments of horror created by the "monster-functions," such as continuous functions with no derivative at any point, that challenged the mathematicians' prevailing intuitions and seemed to have no other purpose than to steal from them their confidence in their own reasoning. The evolution of the concept of function in this early phase was marked by two aspects: a concern with coherence and another with generality. The discussion, however, did not proceed at just an abstract level; it went along the major mathematical problems of the time.

The National Council of Teachers of Mathematics in The Principles and Standards for School Mathematics (2000) states that the secondary school mathematics program must be both broad and deep. (p. 287) They state further that in grades 912, students should encounter new classes of functions Through their high school experiences, they stand to develop deeper understandings of the fundamental mathematical concept of function, (p. 287) Additionally, students need to learn to use a wide range of explicitly and recursively defined functions to model the world around them. Moreover, their understanding of the properties of those functions will give them insights into the phenomena being modeled. (p. 287) This follows in the heels of a reform in the teaching of calculus at the college level and then at the high school level through Advanced Placement courses. The call from the Calculus Reform movement, especially the “Harvard Calculus” group, was to teach the Rule of Three. This was an attempt to get students to realize that there were multiple ways to represent and consider functions: numerically, graphically, and analytically. This are not new ways of studying functions. In fact, as we shall see, they are all quite old. The problem though was that in the preceding time, mathematics had been focusing on one particular representation (analytic) and little time was spent on the other representations, even though they were quite useful in areas where the mathematics was applied. The connections in mathematics were being missed and ignored.

III. RESULTS

Today, mathematics is no longer as exclusively tied to the physical sciences as in the past. It has seen a significant increase in its domains of application, becoming an instrument for the study of the phenomena and situations of biological sciences, human and social sciences, business, communications, engineering, and technology. Mathematics constitutes an essential means of description, explanation, prediction, and control. All the applications of mathematics presuppose the notion of model. A mathematical model is a representation through relations and structures that intends to describe the elements found fundamental in a given situation while deliberately omitting secondary elements. A mathematical model can take several forms, but usually it is constituted by variables, relations among those variables, and their respective rates of change. The process of constructing a mathematical model involves a number of phases, from the situation to a mathematical description and back to the situation again. Several cycles may be required to yield a satisfactory result. Different activities constitute an integral part of this process: · the identification of the elements of the target situation; · the selection of objects, relationships, and so forth, relevant for such purpose; · the idealization of these objects and relations in a form appropriate for mathematical representation; · the choice of the mathematical universe in which the model will be based; · the translation from the situation to this universe; · the establishment of mathematical relationships among the translated objects, accompanied by hypotheses and properties; · the use of mathematical methods to obtain new results and conclusions regarding the model; · the interpretation of these results and conclusions in terms of the original situation; · the evaluation of the model by contrasting it with the situation, checking predicted with observed data, comparing it with other models, or comparing it with theoretical knowledge; and · the modification of the current model or the construction,[8,9,10] if necessary, a new one (Niss, 1987). Dynamical systems are among the most frequently mathematically modelled phenomena. In these models, the fundamental variables indicate the states of the system, representing quantities known as "accumulations" that may increase or decrease, such as distance to origin, mass, kinetic energy, volume of a liquid, number of living organisms, and so forth. Equations involving the rates of change of these quantities indicate their variations over time. Models can be of other kinds, though. In spatial distribution models, how different objects or quantities are distributed and moving in space is studied. Discrete stochastic models represent sequences of events that take place in time according to some probability functions. In all of these cases, it is always of great interest to study the effects of the different factors that influence the situation. In order to do this efficiently, we need to establish functional relationships involving the parameters that represent such factors and the fundamental state variables of the model. The notion of numerical function as a correspondence between variables is of great importance in the conception and study of mathematical models, whatever their nature. Therefore, it continues to be a key notion in present day applications of mathematics. The Roles of Different Representations Most students arrive at secondary school with many difficulties in abstract thinking. For many, dealing with Cartesian graphs and algebraic expressions is not an easy task. The teaching of functions needs to articulate in a balanced way the three most important forms of representation, namely the numerical, graphical, and algebraic forms. It would be a serious misinterpretation of the historical importance of the analytical and geometrical representations of function to allow it to downplay the role of the numerical aspects on learning about functions, notably tables and computations. In real world situations, concrete numerical values underlie the algebraic expressions and the geometrical curves. Mathematicians of the 17th and 18th centuries spent a large amount of time doing arithmetical operations, looking for patterns and relationships. Newton, for example, filled many pages with long arithmetical computations. Research has found that in the interpretation of functional relationships given through Cartesian graphs, students often resort to numerical reasoning strategies, with which they are more confident (Ponte, 1984). In a pedagogical sense, for these students, numbers are essential primitive entities to which more abstract mathematical concepts need to be frequently referred. It can be argued that many of the difficulties that students experience in school mathematics arise from the pressure to deal predominantly with the more abstract entities, without regard for their natural groundings. To construct and analyze tables, compute numerical values, develop a quantitative sense, and acquire a notion for what are acceptable and unacceptable approximations, are important aspects of the mathematical competence that only may be attained if one can currently and easily deal with concrete numbers, if possible, coming from real life situations. The interpretation of significant features of functions from their Cartesian graphs certainly deserves a well-established place in mathematics curricula. Ideas related to variation, such as increase, decrease, constancy, maximum, and minimum, and with variation in variation, such as fast and slow variation, rate of change, smoothness, continuity, and discontinuity, are better grasped from graphical representations.[7,8,9] To be mathematically literate means to be able to use these concepts to make predictions, interpolate, and extrapolate; to be able to establish relationships among different functions by superimposing graphs; and also, to be able to construct regression curves that approximate relationships for empirically obtained data and have an idea of the degree of association between two variables. Naturally, the work with analytical expressions continues to be important. But, more fundamental than students' ability to manipulate long and intricate expressions correctly, is that students understand the meaning of these expressions in concrete situations. Formulas from geometry, physics, and from other sciences can be taken as examples and explored from diverse viewpoints. Several recent technological developments may have a very significant role in the study of functions. Especially important are graphic calculators and computers with appropriate software such as spreadsheets, graph plotters, and symbol manipulation programs. These technological instruments, wisely used in the mathematics classroom, may help students to develop a deeper sort of mathematical understanding, facilitating the process of conjecturing, and testing and making generalizations. This technology may also give students the necessary power to solve more difficult problems and suggest multiple links among such diverse domains as geometry, algebra, statistics, and real situations and their corresponding mathematical models. The superimposition of graphs of several functions, easily done with a computer or a graphing calculator, enables the study of the influence of several parameters in a family of related functions

A function $f(x)$ is an even function if $f(-x) = f(x)$ for all x. A function $f(x)$ is an odd function if $f(-x) = -f(x)$ for all x. A function may be even or odd or neither. Examples are $f(x) = x^2$, $f(x) = x^3$ and $f(x) = 2x + 1$. Graphically, where even and odd arise is from looking at symmetries of the graph. If we replace a by $-a$ in the argument, then we are looking at what happens across the y-axis. Thus, an even function is symmetric with respect to the y-axis because we get the same value when replacing a by $-a$; $f(-a) = f(a)$. If you can draw its graph for $x > 0$, then you need only reflect that across the y-axis to obtain the other half of its graph. If $f(x)$ is an odd function, then we don't have that property, but instead we have that the value of $f(-a)$ is the opposite of the value of $f(a)$. In other words, we would reflect the graph across the y-axis and then across the x-axis. This is known as reflection through the origin, because (a, f(a)) and (−a, −f(a)) lie on the same line through the origin.

IV. CONCLUSIONS

It is certainly possible to define functions in a very general form, for example, as sets of ordered pairs, emphasizing an algebraic perspective in elementary mathematics. This is not, however, a suitable basis to produce an accessible elementary mathematical theory, rich in interesting results and in significant applications. On the contrary, numerical functions have particularly appealing elementary properties, have simple and intuitive geometrical representations, and are useful to describe many different kinds of situations. Their study allows students to operate from a grounding in former knowledge and in multiple representations of situations with which they are already familiar. Dealing with numerical functions, students may follow, in the most general features, a way similar to that followed by the mathematical community. Numerical functions, taking advantage of the former work done with arithmetic and elementary algebra, provide a new perspective to look at notions previously encountered and establish a bridge to geometry through the Cartesian representation. Therefore, numerical functions are a very appropriate way of introducing the important and more general notion of function. Conclusion There is no single way of looking at functions. Those functions studied in mathematical analysis and used in applications maintain as central the idea of dependency among numerical variables. Those considered in algebra emphasize the notion of relationship and those studied in mathematical logic and computer science mostly value the algorithmic aspects. In pedagogical terms, it seems appropriate to present functions as correspondences between numerical sets. The "well-behaved" examples, for which there is an analytical expression or a simple rule, must be clearly emphasized in school mathematics. The identification of functions with analytical expressions, which some students will be likely to make, does not need to be seen as a "big mistake." It is, as history tells us, a natural and fecund association that does not give rise to any particular difficulties at an elementary level. Those few students who will need a more sophisticated concept, will certainly have further opportunities to come back to the notion of function and enlarge and refine it. School mathematics has long focused on algebraic manipulation; however, the ability to deal with algebraic expressions is not enough to solve real problems. [10]Students need to be provided with opportunities for practice and reflection on solving significant problems. In this respect, technology may play an important educational role, changing the focus from mechanical and repetitive processes to the comprehension of algebra and calculus as instruments that enable the modelling of real situations. Technology may be used to undertake the manipulations or obtain the solutions within the mathematical models, simplifying the routine aspects of the work and allowing a stronger concentration on the aspects that are truly important in doing and learning mathematics: the comprehension of the meaning of concepts, the formulation of problems, the understanding of their nature, the elaboration of suitable strategies, and their thorough discussion and critical analysis.[11]

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