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Special Functions in Number Theory

Rajendra, Vikas Solanki

Assistant Professor, Department of Mathematics, Dr. Bhim Rao Ambedkar Govt. College, Sri Ganganagar, Rajasthan, India

ABSTRACT: A number of special functions have become important in maths because they arise in frequently encountered situations. Identifying a one-dimensional (1-D) integral as one yielding a special function is almost as good as a straight-out evaluation, in part because it prevents the waste of time that otherwise might be spent trying to carry out the integration. But of perhaps more importance, it connects the integral to the full body of knowledge regarding its properties and evaluation. It is not necessary for every mathematician to know everything about all known special functions, but it is desirable to have an overview permitting the recognition of special functions which can then be studied in more detail if necessary.

KEYWORDS: special functions, number theory, knowledge, evaluations, recognition

I. INTRODUCTION

Functions that occur frequently, and are therefore useful, are often called special functions. In many cases, special functions are solutions of ordinary or partial differential equations or of extremal problems. Traditionally, special functions are divided into two classes: (a) elementary functions and (b) higher transcendental functions.

Furthermore, those higher transcendental functions that are solutions of ordinary differential equations can be classified as algebraically transcendental or transcendentally transcendental. See Rubel [Ru]. An alternative new classification of transcendental functions, with interesting consequences, is introduced in [WZ].

The designation elementary functions is usually applied to the exponential, logarithmic, and trigonometric functions and combinations formed by algebraic operations. Certain other special functions occur so frequently in applied mathematics that they acquire standardized symbols and names—often in honor of a famous mathematician. We will discuss a few examples of interest in physics, chemistry, and engineering, in particular, the special functions named after Bessel, Legendre, Hankel, Laguerre, and Hermite. We already encountered in Article 6 the gamma function, error function, and exponential integral. Special functions are most often solutions of second-order ODEs with variable coefficients, obtained after separation of variables in PDEs. So that you will have seen the names, here is a list of some other special functions you are likely to encounter (although not in this article): Airy functions, Beta functions, Chebyshev polynomials, Elliptic functions, Gegenbauer polynomials, Jacobi polynomials, Mathieu functions, Meijer G-functions, Parabolic cylinder functions, Theta functions, Whittaker functions.[1,2,3]

The use of special functions as a tool for Bayesian computation points to several challenges. Firstly, special functions are not very popular in Bayesian analysis, simply because there are not many research works drawing attention to this field of Mathematics. In addition, relatively a few computational resources are available, any use of special functions are likely be done manually, which requires rather specialised expertise. For instance, when the complexity is increased by adding more than two parameters, we have to deal with multivariate H-functions or some generalisation of H-functions, but as basically nothing is implemented yet, hence heavy work may be required.

As pointed out before, future work may compare the proposed theory with stochastic and approximate methods, and deal with aspects related the human and computational work cost. In particular, we are not sure what slow down the summations, which algorithms are more efficient to sum up infinite series, etc. These computational issues are to be compared with some stochastic and approximate methods such as VB.

Further generalisations of the theory may deal with hierarchical models with several parameters, this additional complication may involve the integration of multivariate H-functions, which still can be done, depending on the hierarchical structure. The computation of the posterior moments may become heavily complicated, mainly when the H-functions involved have poles of higher orders, therefore some efficient computational algorithm for summing up series may be proposed in order to achieve convergence with less computational time.

we shall review some of those aspects of the theory of special functions that are relevant for Geometric Function Theory, especially for conformal and quasiconformal maps. We restrict ourselves mainly to most important special functions, and briefly mention contexts where these functions occur. We do not attempt to give the results in their most general form, but rather our goal is to provide formulations which are easy to understand and adequate for many applications. As a rule, we have avoided giving proofs of results which are classical and available in the well-known monographs of the field – in such cases we have indicated references for the proof and possible generalizations or related results.

One of our favorite references of the field is [Ra] which covers all the aforementioned topics in a lucid and economical style. Another standard reference of the subject is [WW], a treatise that in its first edition was published at the end of 19th century. Some of the encyclopedic works of the field are [Bat1,Bat2,Bat3,AS]. The monumental three-volume series [Bat1,Bat2,Bat3] was, at its time of publication in the early 1950s, a state-of-the-art survey of the field and it is still the most comprehensive treatment of special functions. [AS] and [Bat1,Bat2,Bat3] are among the most frequently cited articles on special functions. The reference [AS] is a most useful collection of formulas, graphs, and tabular data of numerical values of special functions. For a short excellent outline of the history of special functions

A function indicates dependence of one varying quantity on another. The focus of this article is on basic aspects of functions and some special functions with their applications. First, domain, codomain, and range of functions along with independent and dependent variables are introduced. Then special functions are discussed, including increasing, nonincreasing, nondecreasing, and decreasing functions as well as piecewise defined functions, real-valued, and integer-valued functions. After presenting one-to-one and onto functions, one-to-one correspondence and invertible[4,5,6] functions are detailed. Finally, the focus turns toward compositions of functions.

A function h in $b\mathscr{E}_+$ is special and co-special if and only if there is a real number c such that
$Uch>1\otimes mandU\hat{}ch>1\otimes m.$
Proof
The "if" part is obvious. To prove the converse, we start with P and $P\hat{}$ quasi-compact. Since the constant functions are special there is, by virtue of ch. 6, Lemma 4.7, a function f > 0 and a constant a < 1 such that $U_a > 1 \otimes fm$. For a′ < a, we have $U_{a'} > (a - a')\, U_a U_{a'}$, and therefore, for all $g \in b\mathscr{E}_+$ and x ∈ E,
$Ua'g(x)>(a-a')UaUa'g(x) \quad >(a-a')\ \langle f,Ua'g\rangle\ =(a-a')\ \langle U\hat{}a'f,g\rangle$.
Since $P\hat{}$ is quasi-compact, there is, by Lemma 1.2, a number a_0 such that $a' < a_0$ implies $a'U\hat{}a'f\geq k>0$; thus
$Ua'g(x)>k(a-a')a'm(g).$
This proves that, for a′ sufficiently small, $U_{a'} > 1 \otimes m$. Symmetrically, one can find a″ such that $U\hat{}a''>1\otimes m$ Then, taking c = a′ ∧ a″, we get the desired result for constant functions.
Let us now shift to the general case and suppose h > 0. The T.P.'s Q = $U_h I_h$ and $Q^{-}=U\hat{}hIh$ are quasi-compact and in duality with respect to the measure hm. Since $UcQ=UchIh$, the first part of the proof implies that for c sufficiently small,
$UchIh>1\otimes h m$and$U\hat{}chIh>1\otimes hm$;
hence
$Uch>1\otimes m$and$U\hat{}ch>1\otimes m.$
Now, any special and co-special function h is majorized by a strictly positive special function k and by a strictly positive co-special function $k\hat{}$. The function $k\wedge k\hat{}$ is strictly positive, special and co-special, and majorizes h. Since U_h increases when h decreases, the proof is complete.
The importance of functions h such that $U_h > 1 \otimes m$ was shown in ch. 6 §5. The above results lead one to think that it is exactly those functions which are both special and co-special. The following results lead in that direction.

II. DISCUSSION

We proceed with the study of Harris chains. Our goal is twofold: we want to build a potential theory suitable for the recurrent case – in particular we want to solve the Poisson equations – this will be initiated in the next section and achieved in chs. 8 and 9; on the other hand we want to show a general quotient limit theorem generalizing ch. 4, Exercise 4.10, and this will be done in § 6. Throughout this study the class of functions to be defined below will play a prominent role.

Definition 4.1
A function $f \in \mathscr{E}_+$ is said to be special if for every non-negligible function $g \in U_+$, the function $U_g(f)$ is bounded. A set $A \in \mathscr{E}$ is called special if its characteristic function 1_A is special.

Proposition 4.2
The set S of special functions is a convex hereditary sub-cone of $L^1_+(m)$.
Proof[7,8,9]
Let $g \in U_+$ and $0 < m(g) < \infty$ by ch. 3, Proposition 2.9,
$m(f)=m(gUg(f))\leqslant m(g)\|Ug(f)\|<\infty$.
There exists a special function which is strictly positive and continuous.
Proof
Let h be a strictly positive special function. There exists a number a such that $\infty > m(\{h \geq a\}) > 0$. The set $\{h \geq a\}$ contains a compact set K with $m(K) > 0$, and K is special.
For $0 < c < 1$, the function $U_c(\cdot, K)$ is upper-semi-continuous, strictly positive and special. The space E is the union of the closed sets $\{U_c(\cdot, K)\, n^{-1}\}$. By Baire's theorem one of these sets has a non-empty interior; there exists a non-zero function φ in $CK+$ that has support in this set and so is special. By Condition 4.1 we have $m(\varphi) > 0$, and therefore $U_c\varphi$ is special, continuous and strictly positive.

Special functions have also been traditionally significant in both algebraic geometry and integrable systems. Within the examples presented, elliptic functions gave rise to surprisingly sophisticated theories. The 1-wave solution encountered in the introduction, $u = 2\wp + \text{const.}$ in the limit when one or both periods of the Weierstrass function go to zero, becomes exponential or rational, respectively. The higher-genus analogs give rise to solitons, or rational solutions. On the other hand, the KP solutions which are doubly periodic in the x variable ("elliptic solitons") were classified by Krichever (cf. Dubrovin et al. (2001)), as forming an ACI Hamiltonian system ("elliptic Calogero–Moser"), which, 25 years later, is still generating important work, with Hamiltonian

$H=\sum i=1np12+12\sum i\neq j\wp(qi-qj)$
(where $\wp$ is the Weierstrass function of a lattice L with associated elliptic curve $X=C/L, q\in X$ the origin) and $u=2\sum i=1n\wp(x-xi(t2,t3,\ldots))$ is a solution of the KP hierarchy for suitable time flows t_j of the system ($t_1 = x$) and KP Baker function
$\psi(x;\alpha)=\sigma(\alpha-x)\sigma(\alpha)-\sigma(x)\mathrm{e}(\varsigma(\alpha)\cdot x)$
The associated spectral curves have been classified in moduli by Treibich and Verdier (cf. Treibich (2001)); Krichever produced a two-field model as well as a universal Poisson structure for the system; Donagi and Markman (1996) realized it as a generalized Hitchin system.

More classically, elliptic potentials were the subject of much study, in particular by Lamé and Hermite in the nineteenth century and Ince in the twentieth; a sample result due to Ince makes one feel like Alice in Wonderland, who "knelt down and looked along the passage into the loveliest garden you ever saw": the Lamé operator $L = -\partial^2 + a(a + 1)\wp(x - x_0)$ with real, smooth potential is finite gap (namely, almost all the periodic eigenvalues are double) iff $a\in Z$ (if a is positive the number of gaps is a). A generalization to several variables (due to Chalykh and Veselov),
$L=-\Delta+\sum\alpha\in R+g\alpha\wp(\langle\alpha,x\rangle)$
where R_+ is the set of positive roots for a simple complex Lie algebra of rank n, $\langle -,- \rangle$ is some scalar product in Rn, invariant under the action of the Weyl group, and $g_\alpha = m_\alpha(m_\alpha + 1)\langle\alpha, \alpha\rangle$ for some $m\alpha\in Z$, provides one of the few known examples of quantum completely integrable rings of differential operators in several variables. Roughly speaking, this means that the centralizer of L contains n operators with functionally independent symbols, where n is the number of variables.

What is more, Chalykh et al. (2003) combine differential Galois theory and elliptic function theory to characterize (under some mild assumptions) the generalized Lamé operators that are algebraically completely integrable: the differential Galois group of the solutions is abelian.[10,11,12]

III. RESULTS

Special functions are particular mathematical functions that have more or less established names and notations due to their importance in mathematical analysis, functional analysis, geometry, physics, or other applications.
The term is defined by consensus, and thus lacks a general formal definition, but the list of mathematical functions contains functions that are commonly accepted as special.

Tables of special functions
Many special functions appear as solutions of differential equations or integrals of elementary functions. Therefore, tables of integrals[1] usually include descriptions of special functions, and tables of special functions[2] include most important integrals; at least, the integral representation of special functions. Because symmetries of differential

equations are essential to both physics and mathematics, the theory of special functions is closely related to the theory of Lie groups and Lie algebras, as well as certain topics in mathematical physics.

Symbolic computation engines usually recognize the majority of special functions.

Evaluation of special functions

Most special functions are considered as a function of a complex variable. They are analytic; the singularities and cuts are described; the differential and integral representations are known and the expansion to the Taylor series or asymptotic series are available. In addition, sometimes there exist relations with other special functions; a complicated special function can be expressed in terms of simpler functions. Various representations can be used for the evaluation; the simplest way to evaluate a function is to expand it into a Taylor series. However, such representation may converge slowly or not at all. In algorithmic languages, rational approximations are typically used, although they may behave badly in the case of complex argument(s).

History of special functions

Classical theory

While trigonometry and exponential functions were systematized and unified by the eighteenth century, the search for a complete and unified theory of special functions has continued since the nineteenth century. The high point of special function theory in 1800–1900 was the theory of elliptic functions; treatises that were essentially complete, such as that of Tannery and Molk,[3] expounded all the basic identities of the theory using techniques from analytic function theory (based on complex analysis). The end of the century also saw a very detailed discussion of spherical harmonics.

Changing and fixed motivations

While pure mathematicians sought a broad theory deriving as many as possible of the known special functions from a single principle, for a long time the special functions were the province of applied mathematics. Applications to the physical sciences and engineering determined the relative importance of functions. Before electronic computation, the importance of a special function was affirmed by the laborious computation of extended tables of values for ready look-up, as for the familiar logarithm tables. (Babbage's difference engine was an attempt to compute such tables.) For this purpose, the main techniques are:

- numerical analysis, the discovery of infinite series or other analytical expressions allowing rapid calculation; and
- reduction of as many functions as possible to the given function.

More theoretical questions include: asymptotic analysis; analytic continuation and monodromy in the complex plane; and symmetry principles and other structural equations.

Twentieth century

The twentieth century saw several waves of interest in special function theory. The classic Whittaker and Watson (1902) textarticle[4] sought to unify the theory using complex analysis; the G. N. Watson tome A Treatise on the Theory of Bessel Functions pushed the techniques as far as possible for one important type, including asymptotic results.

The later Bateman Manuscript Project, under the editorship of Arthur Erdélyi, attempted to be encyclopedic, and came around the time when electronic computation was coming to the fore and tabulation ceased to be the main issue.[13,14,15]

Contemporary theories

The modern theory of orthogonal polynomials is of a definite but limited scope. Hypergeometric series, observed by Felix Klein to be important in astronomy and mathematical physics,[5] became an intricate theory, in need of later conceptual arrangement. Lie groups, and in particular their representation theory, explain what a spherical function can be in general; from 1950 onwards substantial parts of classical theory could be recast in terms of Lie groups. Further, work on algebraic combinatorics also revived interest in older parts of the theory. Conjectures of Ian G. Macdonald helped to open up large and active new fields with the typical special function flavour. Difference equations have begun to take their place besides differential equations as a source for special functions.

Special functions in number theory

In number theory, certain special functions have traditionally been studied, such as particular Dirichlet series and modular forms. Almost all aspects of special function theory are reflected there, as well as some new ones, such as came out of the monstrous moonshine theory.

Special functions of matrix arguments
Analogues of several special functions have been defined on the space of positive definite matrices, among them the power function which goes back to Atle Selberg,[6] the multivariate gamma function,[7] and types of Bessel functions.[8] The NIST Digital Library of Mathematical Functions has a section covering several special functions of matrix arguments.[9]

The first topic covered in [1] is built around the Gauss hypergeometric function $2F1$, the Kummer hypergeometric function $1F1$, and their generalization rFs. Summation formulae contiguous to the q-Kummer summation theorems are established. They are relevant to the theory of partitions. May be it is a good place to quote other works where hypergeometric function $2F1$ plays a role in number theory with the concepts of `dessins d'enfants' and the Painlevé VI equation [2,3]. These references enable the placing of the subject in a broad perspective that links algebraic number theory (through the Belyi function of dessins), symmetry (through the monodromy group around the singularities), and Painlevé transcendents. Hypergeometric Bernoulli numbers and polynomials connect to the hypergeometric function $1F1$ which is the subject of [4]. The later paper explores hypergeometric polynomials with Lagrange polynomials in several variables and other polynomials. In a similar vein, ref. [5] shows the role of generating functions for parametrically generalized polynomials. They are related to the combinatorial numbers, the Bernoulli polynomials and numbers, the Euler polynomials and numbers, etc.
The second topic is about analytic number theory, the Riemann zeta function (s) and other functions found in number theory. In [6], the authors prove an asymptotic formula for the sum of the values of the periodic zeta-function at the non-trivial zeros of (s). Their proof holds unconditionally (irrespective of the Riemann hypothesis). Ref. [7] focuses upon the Riemann hypothesis and random walks. The method allows the calculation of critical zeros of (s) at very large order, such as the 10100-th zero. Again in the context of analytic number theory, the main aim of article [8] is to investigate some interesting symmetric identities for the Dirichlet-type multiple (p,q)-L function in relation to Euler polynomials and generalized Euler polynomials. Ref. [9] defines p-adic and q-Dedekind type sums. Dedekind reciprocity laws and their symmetries are a cornerstone of this study. Then, ref. [10] defines a new form of Carlitz's type degenerate twisted $(p,)$-Euler numbers and polynomials and study some theories of the Carlitz's type degenerate twisted $(p,)$-Euler numbers and polynomials. Finally, paper [11] investigates certain identities associated with $(p,)$-binomial coefficients and $(p,)$-Stirling polynomials of the second kind.

The note on a triple integral [12], still about the second topic, deserves a special mention due to its originality and the perspectives it offers in mathematical physics. A general integral theorem is developed in terms of the Lerch function, Hurwitz zeta function, polylogarithm function, and the Riemann zeta function. Ref. [13] is a survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics. The later two papers can be studied in parallel to contemplate the richness of the connection between analytical number theory and special functions.

A third topic of recurrent interest is about Fibonacci and Lucas numbers, and the Golden ratio. Refs. [14,15,16,17,18] illustrate their applications to chemistry, biology, physics, social sciences, anthropology, etc.[16,17,18]

IV. CONCLUSION

Special functions arise in many problems of pure and applied mathematics, mathematical statistics, physics, and engineering. This article provides an up-to-date overview of numerical methods for computing special functions and discusses when to use these methods depending on the function and the range of parameters. Not only are standard and simple parameter domains considered, but methods valid for large and complex parameters are described as well.

The first part of the article (basic methods) covers convergent and divergent series, Chebyshev expansions, numerical quadrature, and recurrence relations. Its focus is on the computation of special functions; however, it is suitable for general numerical courses. Pseudoalgorithms are given to help students write their own algorithms. In addition to these basic tools, the authors discuss other useful and efficient methods, such as methods for computing zeros of special functions, uniform asymptotic expansions, Padé approximations, and sequence transformations. The article also provides specific algorithms for computing several special functions (like Airy functions and parabolic cylinder functions, among others).[18]

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