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Recent Developments on Summability and Approximation Theory and It's Application

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ABSTRACT: In mathematics, approximation theory is concerned with how functions can best be approximated with simpler functions, and with quantitatively characterizing the errors introduced thereby. What is meant by best and simpler will depend on the application.[15]

A closely related topic is the approximation of functions by generalized Fourier series, that is, approximations based upon summation of a series of terms based upon orthogonal polynomials.

One problem of particular interest is that of approximating a function in a computer mathematical library, using operations that can be performed on the computer or calculator (e.g. addition and multiplication), such that the result is as close to the actual function as possible. This is typically done with polynomial or rational (ratio of polynomials) approximations.

The objective is to make the approximation as close as possible to the actual function, typically with an accuracy close to that of the underlying computer's floating point arithmetic. This is accomplished by using a polynomial of high degree, and/or narrowing the domain over which the polynomial has to approximate the function. Narrowing the domain can often be done through the use of various addition or scaling formulas for the function being approximated. Modern mathematical libraries often reduce the domain into many tiny segments and use a low-degree polynomial for each segment.

In mathematics, a series is, roughly speaking, the operation of adding infinitely many quantities, one after the other, to a given starting quantity.[1] The study of series is a major part of calculus and its generalization, mathematical analysis. Series are used in most areas of mathematics, even for studying finite structures (such as in combinatorics) through generating functions. In addition to their ubiquity in mathematics, infinite series are also widely used in other quantitative disciplines such as physics, computer science, statistics and finance.

For a long time, the idea that such a potentially infinite summation could produce a finite result was considered paradoxical. This paradox was resolved using the concept of a limit during the 17th century. Zeno's paradox of Achilles and the tortoise illustrates this counterintuitive property of infinite sums: Achilles runs after a tortoise, but when he reaches the position of the tortoise at the beginning of the race, the tortoise has reached a second position; when he reaches this second position, the tortoise is at a third position, and so on. Zeno concluded that Achilles could never reach the tortoise, and thus that movement does not exist. Zeno divided the race into infinitely many sub-races, each requiring a finite amount of time, so that the total time for Achilles to catch the tortoise is given by a series. The resolution of the paradox is that, although the series has an infinite number of terms, it has a finite sum, which gives the time necessary for Achilles to catch up with the tortoise.

KEYWORDS: mathematics, approximation, summability, generative, paradox, potentially, statistics

I.INTRODUCTION

Once the domain (typically an interval) and degree of the polynomial are chosen, the polynomial itself is chosen in such a way as to minimize the worst-case error. That is, the goal is to minimize the maximum value of , where P(x) is the approximating polynomial, f(x) is the actual function, and x varies over the chosen interval.[16] For well-behaved functions, there exists an Nth-degree polynomial that will lead to an error curve that oscillates back and forth between and a total of N+2 times, giving a worst-case error of[1] . It is seen that there exists an Nth-degree polynomial that can interpolate N+1 points in a curve. That such a polynomial is always optimal is asserted by the equioscillation theorem. It is possible to make contrived functions f(x) for which no such polynomial exists, but these occur rarely in practice.[2]

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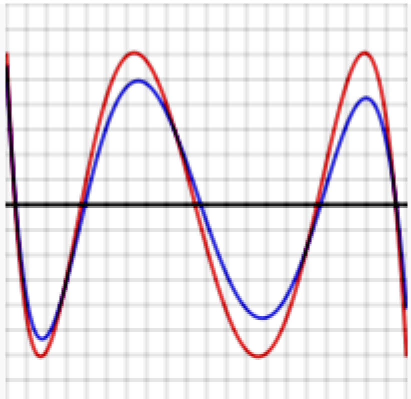
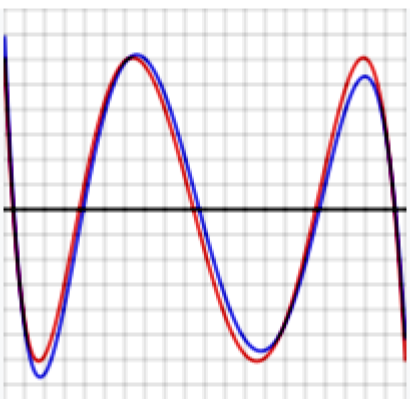
For example, the graphs shown to the right show the error in approximating log(x) and exp(x) for N = 4. The red curves, for the optimal polynomial, are level, that is, they oscillate between and exactly. In each case, the number of extrema is N+2, that is, 6[17]. Two of the extrema are at the end points of the interval, at the left and right edges of the graphs.[3]

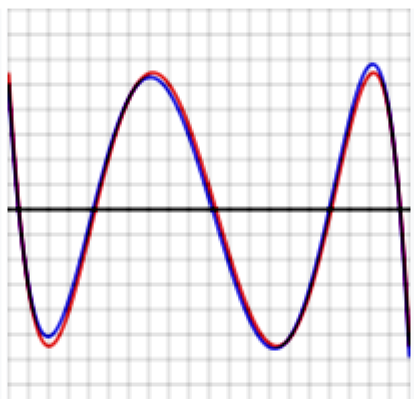


Error P(x) − f(x) for level polynomial (red), and for purported better polynomial (blue)

To prove this is true in general, suppose P is a polynomial of degree N having the property described, that is, it gives rise to an error function that has N + 2 extrema, of alternating signs and equal magnitudes. The red graph to the right shows what this error function might look like for N = 4. Suppose Q(x) (whose error function is shown in blue to the right) is another N-degree polynomial that is a better approximation to f than P.[4]



Error between optimal polynomial and log(x) (red), and Chebyshev approximation and log(x) (blue) over the interval [2, 4]. Vertical divisions are 10^{-5}. Maximum error for the optimal polynomial is 6.07×10^{-5}.



Error between optimal polynomial and exp(x) (red), and Chebyshev approximation and exp(x) (blue) over the interval [−1, 1]. Vertical divisions are 10^{-4}. Maximum error for the optimal polynomial is 5.47×10^{-4}.

II.DISCUSSION

Uniform convergence is desirable for a series because many properties of the terms of the series are then retained by the limit. For example, if a series of continuous functions converges uniformly, then the limit function is also continuous. [5]Similarly, if the f_n are integrable on a closed and bounded interval I and converge uniformly, then the series is also integrable on I and can be integrated term-by-term. Tests for uniform convergence include the Weierstrass' M-test, Abel's uniform convergence test, Dini's test, and the Cauchy criterion.[6]

More sophisticated types of convergence of a series of functions can also be defined. In measure theory, for instance, a series of functions converges almost everywhere if it converges pointwise except on a certain set of measure zero[18]. Other modes of convergence depend on a different metric space structure on the space of functions under consideration. [7]

The error graph does indeed take on the values at the six test points, including the end points, but that those points are not extrema. If the four interior test points had been extrema (that is, the function P(x)f(x) had maxima or minima there), the polynomial would be optimal.[8]

The second step of Remez's algorithm consists of moving the test points to the approximate locations where the error function had its actual local maxima or minima. For example, one can tell from looking at the graph that the point at −0.1[19] should have been at about −0.28. The way to do this in the algorithm is to use a single round of Newton's method. Since one knows the first and second derivatives of P(x) − f(x), [20]one can calculate approximately how far a test point has to be moved so that the derivative will be zero.[9]

Calculating the derivatives of a polynomial is straightforward. One must also be able to calculate the first and second derivatives of f(x).

Estimation theory is a branch of statistics that deals with estimating the values of parameters based on measured empirical data that has a random component. The parameters describe an underlying physical setting in such a way that their value affects the distribution of the measured data. An estimator attempts to approximate the unknown parameters using the measurements. In estimation theory, two approaches are generally considered:[1]

- The probabilistic approach (described in this article) assumes that the measured data is random with probability distribution dependent on the parameters of interest
- The set-membership approach assumes that the measured data vector belongs to a set which depends on the parameter vector.
  - For example, it is desired to estimate the proportion of a population of voters who will vote for a particular candidate. That proportion is the parameter sought; the estimate is based on a small random sample of voters. Alternatively, it is desired to estimate the probability of a voter voting for a particular candidate, based on some demographic features, such as age.[10]
  - Or, for example, in radar the aim is to find the range of objects (airplanes, boats, etc.) by analyzing the two-way transit timing of received echoes of transmitted pulses. Since the reflected pulses are unavoidably embedded in electrical noise, their measured values are randomly distributed, so that the transit time must be estimated.[22]
  - As another example, in electrical communication theory, the measurements which contain information regarding the parameters of interest are often associated with a noisy signal.[21]

III.RESULTS

Approximation arises naturally in scientific experiments. The predictions of a scientific theory can differ from actual measurements. This can be because there are factors in the real situation that are not included in the theory. For example, simple calculations may not include the effect of air resistance.[23] Under these circumstances, the theory is an approximation to reality. Differences may also arise because of limitations in the measuring technique. In this case, the measurement is an approximation to the actual value.[11]

The history of science shows that earlier theories and laws can be approximations to some deeper set of laws. Under the correspondence principle, a new scientific theory should reproduce the results of older, well-established, theories in those domains where the old theories work.[7] The old theory becomes an approximation to the new theory.[24]

Some problems in physics are too complex to solve by direct analysis, or progress could be limited by available analytical tools. Thus, even when the exact representation is known, an approximation may yield a sufficiently accurate solution while reducing the complexity of the problem significantly. Physicists often approximate the shape of the Earth as a sphere even though more accurate representations are possible, because many physical characteristics (e.g., gravity) are much easier to calculate for a sphere than for other shapes.[12]

Approximation is also used to analyze the motion of several planets orbiting a star. This is extremely difficult due to the complex interactions of the planets' gravitational effects on each other.[8] An approximate solution is effected by performing iterations. In the first iteration, the planets' gravitational interactions are ignored, and the star is assumed to be fixed. If a more precise solution is desired, another iteration is then performed, using the positions and motions of the planets as identified in the first iteration, but adding a first-order gravity interaction from each planet on the others. This process may be repeated until a satisfactorily precise solution is obtained.[25]

The use of perturbations to correct for the errors can yield more accurate solutions. Simulations of the motions of the planets and the star also yields more accurate solutions.[26]

The most common versions of philosophy of science accept that empirical measurements are always approximations — they do not perfectly represent what is being measured.[27]

In mathematics, summation is the addition of a sequence of any kind of numbers, called addends or summands; the result is their sum or total. Beside numbers, other types of values can be summed as well: functions, vectors, matrices, polynomials and, in general, elements of any type of mathematical objects on which an operation denoted "+" is defined.[28]

Summations of infinite sequences are called series. They involve the concept of limit, and are not considered in this article[13]

IV.CONCLUSIONS

In mathematics, the Poisson summation formula is an equation that relates the Fourier series coefficients of the periodic summation of a function to values of the function's continuous Fourier transform. [29]Consequently, the periodic summation of a function is completely defined by discrete samples of the original function's Fourier transform. [30]And conversely, the periodic summation of a function's Fourier transform is completely defined by discrete samples of the original function. [31,32,33]The Poisson summation formula was discovered by Siméon Denis Poisson and is sometimes called Poisson resummation.[14]

REFERENCES

1. The Concise Oxford Dictionary, Eighth edition 1990, ISBN 0-19-861243-5
2. ^ Longman Dictionary of Contemporary English, Pearson Education Ltd 2009, ISBN 978 1 4082 1532 6
3. ^ "Numerical Computation Guide". Archived from the original on 2016-04-06. Retrieved 2013-06-16.
4. ^ "Approximately Equal -- from Wolfram MathWorld". Wolfram MathWorld. Retrieved 2020-11-22.
5. ^ "Mathematical Operators – Unicode" (PDF). Retrieved 2013-04-20.
6. ^ D & D Standard Oil & Gas Abbreviator. PennWell. 2006. p. 366. ISBN 9781593701086. Retrieved May 21, 2020. ≐ approaches a limit
7. ^ Encyclopædia Britannica
8. ^ The three body problem
9. ^ European Commission, Guide to the Approximation of European Union Environmental Legislation, last updated 2 August 2019, accessed 15 November 2019
10. ^ EUR-Lex, Directive (EU) 2015/2436 of the European Parliament and of the Council of 16 December 2015 to approximate the laws of the Member States relating to trade marks (recast) (Text with EEA relevance), published 23 December 2015, accessed 15 November 2019
11. "Summation Notation". www.columbia.edu. Retrieved 2020-08-16.
12. ^ Handbook of Discrete and Combinatorial Mathematics, Kenneth H. Rosen, John G. Michaels, CRC Press, 1999, ISBN 0-8493-0149-1.
13. ^ "Calculus I - Summation Notation". tutorial.math.lamar.edu. Retrieved 2020-08-16.
14. ^ Burton, David M. (2011). The History of Mathematics: An Introduction (7th ed.). McGraw-Hill. p. 414. ISBN 978-0-07-338315-6.
15. ^ Leibniz, Gottfried Wilhelm (1899). Gerhardt, Karl Immanuel (ed.). Der Briefwechsel von Gottfried Wilhelm Leibniz mit Mathematikern. Erster Band. Berlin: Mayer & Müller. p. 154.
16. ^ Cajori (1929), pp. 181-182.
17. ^ Cajori (1929), p. 61.
18. ^ Euler, Leonhard (1755). Institutiones Calculi differentialis (in Latin). Petropolis. p. 27.
19. ^ Lagrange, Joseph-Louis (1867–1892). Oeuvres de Lagrange. Tome 3 (in French). Paris. p. 451.
20. ^ Mémoires de l'Académie royale des sciences de l'Institut de France pour l'année 1825, tome VIII (in French). Paris: Didot. 1829. pp. 581-622.

21. ^ Fourier, Jean-Baptiste Joseph (1888–1890). Oeuvres de Fourier. Tome 2 (in French). Paris: Gauthier-Villars. p. 149.
22. Pinsky, M. (2002), Introduction to Fourier Analysis and Wavelets., Brooks Cole, ISBN 978-0-534-37660-4
23. ^ Zygmund, Antoni (1968), Trigonometric Series (2nd ed.), Cambridge University Press (published 1988), ISBN 978-0-521-35885-9
24. ^ Córdoba, A., "La formule sommatoire de Poisson", Comptes Rendus de l'Académie des Sciences, Série I, 306: 373–376
25. ^ Hörmander, L. (1983), The analysis of linear partial differential operators I, Grundl. Math. Wissenschaft., vol. 256, Springer, doi:10.1007/978-3-642-96750-4, ISBN 3-540-12104-8, MR 0717035
26. ^ Oppenheim, Alan V.; Schafer, Ronald W.; Buck, John R. (1999). Discrete-time signal processing (2nd ed.). Upper Saddle River, N.J.: Prentice Hall. ISBN 0-13-754920-2. samples of the Fourier transform of an aperiodic sequence x[n] can be thought of as DFS coefficients of a periodic sequence obtained through summing periodic replicas of x[n].
27. ^ Deitmar, Anton; Echterhoff, Siegfried (2014), Principles of Harmonic Analysis, Universitext (2 ed.), doi:10.1007/978-3-319-05792-7, ISBN 978-3-319-05791-0
28. ^ Grafakos, Loukas (2004), Classical and Modern Fourier Analysis, Pearson Education, Inc., pp. 253–257, ISBN 0-13-035399-X
29. ^ Stein, Elias; Weiss, Guido (1971), Introduction to Fourier Analysis on Euclidean Spaces, Princeton, N.J.: Princeton University Press, ISBN 978-0-691-08078-9
30. ^ Katznelson, Yitzhak (1976), An introduction to harmonic analysis (Second corrected ed.), New York: Dover Publications, Inc, ISBN 0-486-63331-4
31. ^ Kinayman, Noyan; Aksun, M. I. (1995). "Comparative study of acceleration techniques for integrals and series in electromagnetic problems". Radio Science. 30 (6): 1713–1722. doi:10.1029/95RS02060. hdl:11693/48408.
32. ^ Woodward, Philipp M. (1953). Probability and Information Theory, with Applications to Radar. Academic Press, p. 36.
33. ^ H. M. Edwards (1974). Riemann's Zeta Function. Academic Press, pp. 209–11. ISBN 0-486-41740-9.
34. ^ Cohn, Henry; Elkies, Noam (2003), "New upper bounds on sphere packings I", Ann. of Math., 2, 157 (2): 689–714, arXiv:math/0110009, doi:10.4007/annals.2003.157.689, MR 1973059