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The Spectrum of Zero Divisor Graphs and their Complements

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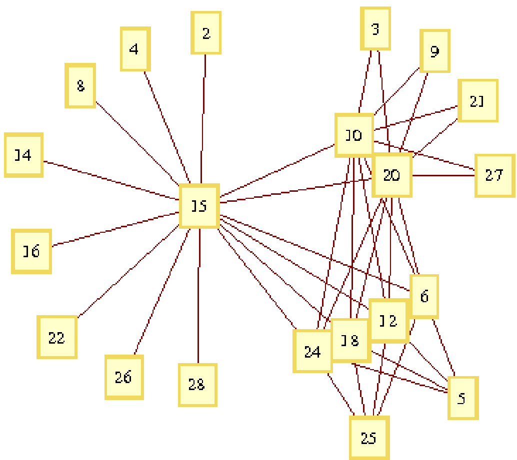
ABSTRACT: In mathematics, and more specifically in combinatorial commutative algebra, a zero-divisor graph is an undirected graph representing the zero divisors of a commutative ring. It has elements of the ring as its vertices, and pairs of elements whose product is zero as its edges.[1] There are two variations of the zero-divisor graph commonly used. In the original definition of Beck (1988), the vertices represent all elements of the ring.[2] In a later variant studied by Anderson & Livingston (1999), the vertices represent only the zero divisors of the given ring.[3]

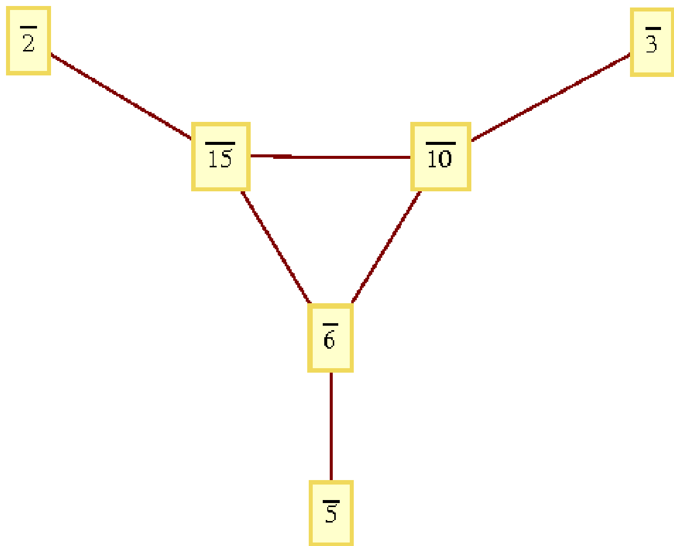
KEYWORDS-mathematics,algebra,zero,divisor,graphs,commutative,ring,complements

I. INTRODUCTION

In the version of the graph that includes all elements, 0 is a universal vertex, and the zero divisors can be identified as the vertices that have a neighbor other than 0. Because it has a universal vertex, the graph[1,2,3] of all ring elements is always connected and has diameter at most two. The graph of all zero divisors is non-empty for every ring that is not an integral domain. It remains connected, has diameter at most three,[3] and (if it contains a cycle) has girth at most four.[4][5]



(a) $\Gamma(\mathbb{Z}_{30})$



(b) $\Gamma_c(\mathbb{Z}_{30})$

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We survey the research conducted on zero divisor graphs, with a focus on zero divisor graphs determined by equivalence classes of zero divisors of a commutative ring R. In particular, we consider the problem of classifying star graphs with any finite number of vertices. We study the pathology of a zero divisor graph in terms of cliques, we investigate when the clique and chromatic numbers are equal, and we show that the girth of a Noetherian ring, if finite, is 3. We also introduce a graph for modules that is useful for studying zero divisor graphs of trivial extensions.

Because so much literature has been written on the topic of various zero divisor graphs, often from very different points of view, we collect here an overview of the material. The terms in bold are defined within the text, while the italicized terms appear in Appendix B. Throughout, and unless otherwise stated, R will be a commutative ring with unity

The zero-divisor graph Γ(R)Γ of a commutative unital ring R is the (simple undirected) graph whose vertex set is all zero divisors of R except 0R0 and two distinct vertices are joined by an edge if and only if their product is zero. In this paper, the structure of strong resolving graph of Γ(R)Γ is completely determined. As an application, we compute the strong metric dimension of Γ(R).[5,7,8]

II. DISCUSSION

Listed below are all the relevant definitions from Graph Theory. A good reference on the subject is [22] and for the material on zero divisor graphs, the papers [10], [2], and [1] provide a good background. (i) A graph is acyclic if it contains no cycles. (ii) A graph is bipartite (respectively, r-partite) if the vertices can be partitioned into two (resp., r) disjoint subsets so that every edge has one vertex in each subset (resp., every edge joins vertices in distinct subsets). (iii) A graph is chordal if every cycle with four or more vertices has a chord, or edge joining two vertices of the cycle that are not adjacent[9]. (iv) The circumference of a graph is the maximum length of a cycle in the graph. If the graph is acyclic, then the circumference is zero. (v) A clique in a graph is a subset of vertices of the graph that are all pairwise adjacent; i.e. a vertex set which induces a complete subgraph. (vi) If a graph G contains a clique of size n and no clique has more than n elements, then the clique number of the graph is said to be n; if the clique size is unbounded, then the clique number is infinite. It is denoted by !.G/. (vii) The closure of a neighborhood of a vertex v in a graph is the neighborhood of v along with v itself; i.e., N .v/ [¹vº. It is denoted by N .v/. (viii) The chromatic number or coloring number of a graph G, denoted .G/, is the minimal number of colors which can be assigned to the vertices of G such that no pair of adjacent vertices has the same color. (ix) A graph is compact if it is a simple connected graph satisfying the property that for every pair of non-adjacent vertices x and y, there is vertex z adjacent to every vertex adjacent to x and/or y. (x) A graph is said to be complete if every vertex in the graph is adjacent to every other vertex in the graph. The notation for a complete graph on n vertices is Kn. (xi) A complete bipartite is a bipartite graph such that every vertex in one partitioning subset is adjacent to every vertex in the other partitioning subset. If the subsets have cardinality m, and n, then this graph is denoted by Km;n. (xii) A complete r-partite graph is an r-partite graph such that every vertex in any partitioning subset is adjacent to every vertex in every other partitioning subset. (xiii) A graph is said to be connected if there is a path between every pair of vertices of the graph.[10]

(xiv) A cut vertex in a connected graph G is a vertex whose removal from the vertex set of G results in a disconnected graph; v is said to separate vertices a and b if every path between the two includes v. (xv) A cycle in a graph is a path of length at least 3 through distinct vertices which begins and ends at the same vertex. (xvi) A cycle graph is an n-gon for some integer n 3. (xvii) The degree of a vertex is the number of vertices adjacent to it. (xviii) The diameter of a connected graph is the supremum of the distances between any two vertices. (xix) A directed graph is a pair .V; E/ of disjoint sets (of vertices and edges) together with two maps initW E ! V and terW E ! V assigning to every edge e an initial vertex init.e/ and a terminal vertex ter.e/. The edge e is said to be directed from init.e/ to ter.e/. (xx) The distance between two vertices v and w in a connected graph is the length of the shortest path between them; if no path exists between a pair of vertices, then the distance is defined to be infinite. (xxi) A vertex is an end if it has degree 1. (xxii) [8,9]The girth of a graph is the length of the shortest cycle in the graph; it is infinite if the graph is acyclic. (xxiii) A graph consists of a set of vertices, a set of edges, and an incident relation, describing which vertices are adjacent (i.e., joined by an edge) to which. (xxiv) Let G D .V; E/ and G0 D .V 0 ; E0 / be two graphs. A homomorphism G ! G0 is a function W V ! V 0 respecting adjacency, that is, such that for all x; y 2 V if xy 2 E, then .x/ .y/ 2 E0 . (xxv) An induced subgraph of a graph G is obtained by taking a subset U of the vertex set of G together with all edges which are incident in G only with vertices belonging to U. (xxvi) Let G D .V; E/ and G0 D .V 0 ; E0 / be two graphs. An isomorphism G Š ! G0 is a bijection W V ! V 0 with xy 2 E if and only if .x/.y/ 2 E0 for all x; y 2 V . (xxvii) The neighborhood of a vertex v in a graph is the set of all vertices adjacent to v. It is denoted by N .v/. [Note that for simple graphs, v … N .v/.][5,7] (xxviii) A non-degenerate star graph is a star graph with at least two vertices. (xxix) A path of length n between two vertices v and w is a finite sequence of vertices u0; u1;:::;un such that v D u0, w D un, and ui1 and ui are adjacent for all 1 i n. (xxx) A graph is perfect if for every induced subgraph, including the graph itself, the chromatic number and clique numbers agree. (xxxi) A graph is planar if it can be drawn in the plane with no crossings of edges. (xxxii) A regular graph is one in which all the vertices have the same degree. (xxxiii) A vertex v in a graph G is said to separate vertices a and b if every path between a and b includes v. (xxxiv) A simple graph is one with no loops on a vertex and no multiple edges between a pair of vertices. (xxxv) A star graph is a complete bipartite graph in which one of the partitioning subsets is a singleton set. The notation for this graph is K1;n.[7,8]

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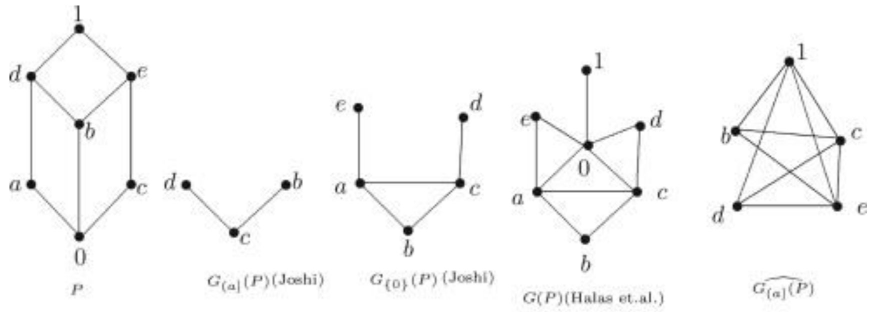
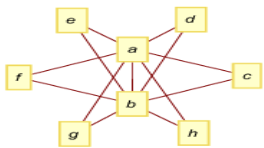
III. RESULTS

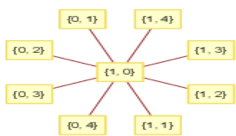


A B C D E

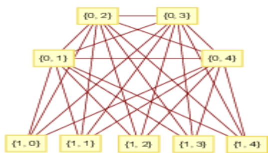
Vertices	Group Type	R	Graph
8	$\mathbb{Z}_9$	$\mathbb{Z}_9^0$	K^8
	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_3^0 \times \mathbb{Z}_3$	Figure 4
	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_3^0 \times \mathbb{Z}_3^0$	K^8
	$\mathbb{Z}_9$	$x\mathbb{Z}[x]/(9x, x^2 - 3x)$	Figure 4
	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$x\mathbb{Z}_3[x]/x^3\mathbb{Z}_3[x]$	Figure 4
9	$\mathbb{Z}_{10}$	$\mathbb{Z}_{10}^0$	K^9
	$\mathbb{Z}_{10}$	$\mathbb{Z}_2^0 \times \mathbb{Z}_5$	Figure 5a
	$\mathbb{Z}_{10}$	$\mathbb{Z}_2 \times \mathbb{Z}_5^0$	Figure 5b
10	$\mathbb{Z}_{11}$	$\mathbb{Z}_{11}^0$	K^{10}



URE 4 8-vertex graph

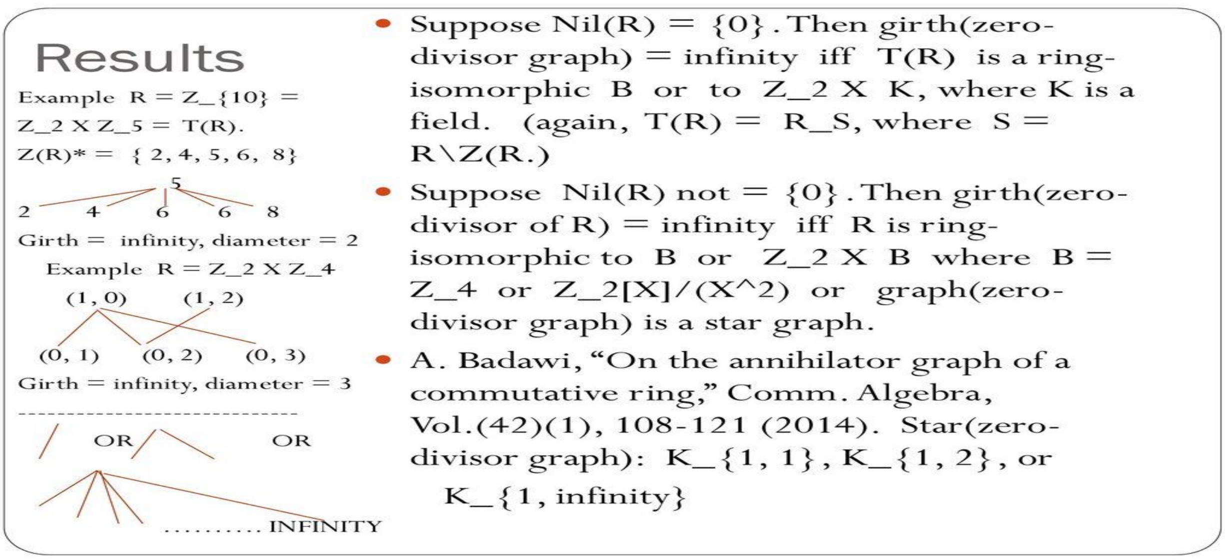


(a)



(b)

IV. CONCLUSION



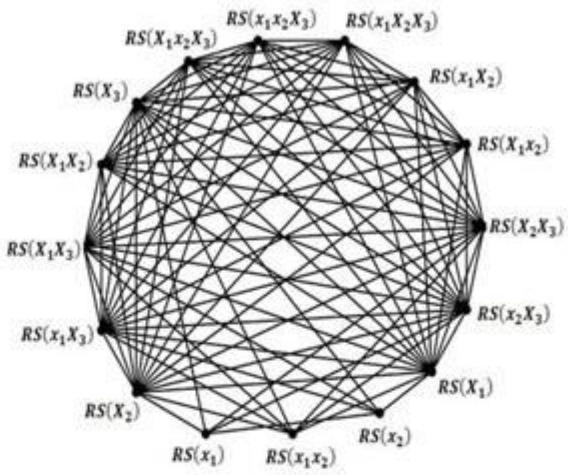


Figure 1: Rough Co-Zero Divisor Graph for n = 3 and m = 2. [10]

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