On Certain Results Associated with Hurwitz-Lerch Zeta Function

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ABSTRACT: In this paper certain results for multiparameter Hurwitz Zeta function are established. Some Euler type transforms associated with Gauss's hyper geometric functions are obtained. A few known and new integrals are also, obtained as special cases of our main results in this paper.

KEYWORDS: Beta functions, Generalized hyper geometric function, Wrights generalized hyper geometric function, Zeta function, Riemann Zeta function, Hurwitz- Lerch Zeta function.

Introduction and Preliminaries:

The familiar general Hurwitz-Lerch zeta function is defined as follows. Srivasatava [5]

$$\phi(z,s,a)=\sum_{l=0}^{\infty}\frac{z^l}{(l+a)^s} \tag{1.1}$$

$(a \in C/Z_0^-; s \in C$ when $|z|<1$; $R(s)>1$ when $|z|=1)$

The integral representation of above defined Hurwitz Lerch zeta function is given by (Erdelyi et al [1] P.27, Equation 1.11(3).

$$\phi(z,s,a)=\frac{1}{\Gamma s}\int_0^{\infty}\frac{t^{s-1}e^{-at}}{1-ze^{-t}}dt \tag{1.2}$$

$\mathrm{Re}(a)>0$, Re(s) >0 when $|z|\le 1(z\ne 1)$, Re(s) >1 when z = 1 at a =0

The Hurwitz-Lerch zeta function contains, as its special cases, the Riemann zeta function $\varsigma(s)$, the Hurwitz zeta function $\varsigma(s,a)$ and the Lerch zeta function $l_s(\xi)$ defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\phi(1,s,1)=\zeta(s,1)(\mathrm{Re}(s)>1)$$

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}=\phi(1,s,a)\left(\mathrm{Re}(s)>1;a\in C\setminus Z_0^-\right),$$

$$l_s(\xi)=\sum_{n=0}^{\infty}\frac{e^{2n\pi i\xi}}{(n+1)^s}=\phi\left(e^{2\pi i\xi},s,1\right)\left(\mathrm{Re}(s)>1;\xi\in R\right)$$

A generalization of the Hurwitz-Lerch Zeta function is also studied by Goyal and Laddha [10] as follows

$$\phi_\mu^*(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_n z^n}{n!(n+a)^s}$$

Where $\mathrm{Re}(\mu)>0$ and $(\mu)_n$ is the Pochhammer symbol with relation $(\mu)_n=\frac{\Gamma(\mu+n)}{\Gamma(\mu)}$ and its integral representation

$$\phi_\mu^*(z,s,a)=\frac{1}{\Gamma s}\int_0^{\infty}t^{s-1}e^{-at}\left(1-ze^{-t}\right)^{-\mu}dt, \tag{1.3}$$

$$\min\{R(a),R(s)\}>0;|z|<1$$

The generalized Hypergeometric function ${}_pF_q$ is defined as follows:

$$ {}_pF_q\left[\begin{matrix}\alpha_1,........,\alpha_p; \\ \beta_1,......,\beta_q;\end{matrix} z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\,........(\alpha_p)_n\,z^n}{(\beta_1)_n\,.........(\beta_q)_n\,n!} \tag{1.4}$$

Where $(\lambda)_n$ is the Pochhammer symbol. The series in (1.4) is known as generalized Gauss series, or simply, the generalized hypergeometric series. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameter $\alpha_1,........,\alpha_p$ and the denominator parameter $\beta_1,......,\beta_q$ may be complex values, provided that

$$\beta_j \neq 0,-1,-2.......;(j=1,.....q)$$

The Fox-Wright generalized hypergeometric function ${}_p\psi_q^*$ which is generalization of the familiar hypergeometric function ${}_pF_q$ defined by Erdelyi et al.[1] as

$$ {}_p\psi_q^*\left[\begin{matrix}(a_1,A_1),.......,(a_p,A_p); \\ (b_1,B_1),.........,(b_q,B_q);\end{matrix} z\right]=\sum_{n=0}^{\infty}\frac{(a_1)_{A_1 n}.....(a_p)_{A_p n}\,z^n}{(b_1)_{B_1 n}......(b_q)_{B_p n}\,n!} \tag{1.5}$$

At $A_i=1(i=1,.....,p)$, $B_j=1(j=1,.....,q)$ it reduces to generalized hypergeometric function ${}_pF_q$.

Further generalization of the above defined Hurwitz-Lerch Zeta function $\phi_\mu(z,s,a)$ and $\phi_\mu^*(z,s,a)$ is recently studied in the following form by Garg et al [7];

$$\phi_{\lambda,\mu,\gamma}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu)_n\quad z^n}{(\gamma)_n\,n!(n+a)^s} \tag{1.6}$$

and $$\phi_{\lambda,\mu,\gamma}(z,s,a)=\frac{1}{\Gamma s}\int_0^\infty t^{s-1}e^{-at}{}_2F_1\left[\begin{matrix}\lambda,\mu \\ \gamma\end{matrix};ze^{-t}\right]dt \tag{1.7}$$

where $\lambda,\mu\in C, a\in C/Z_0^-, s\in C$ when $|z|<1$ $\operatorname{Re}(s+\upsilon-\lambda-\mu)$ when $|z|=1$

Lin and Srivastava [9] also extended the Hurwitz-Lerch zeta function in the following form.

$$\phi_{\mu,\gamma}^{\rho,\sigma}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}\,z^n}{(\gamma)_{\sigma n}(n+a)^s}$$

$$=\frac{1}{\Gamma s}\int_0^\infty t^{s-1}e^{-at}{}_2\psi_1^*\left[\begin{matrix}(\mu,\rho)\ (1,1) \\ (\gamma,\sigma)\end{matrix};ze^{-t}\right]dt \tag{1.8}$$

$$\mu\in C; a,\lambda\in C\,|\,Z_0^-;\ \rho,\sigma\in R^+;\rho<\sigma \text{ when } s,z\in C;\rho=\sigma$$

and $s\in C$ when $|z|<\delta=\rho^{-p}\sigma^{\sigma};\ \rho=\sigma$ and $\operatorname{Re}(s-\mu+v)>1$ when $|z|=\delta$

Bin-Saad [8] established the following generating function for the Hurwitz – Lerch zeta function defined in (1.1)

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\phi(z,s+n,a)t^n=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^{s-\lambda}(n+a-t)^{\lambda}}=V_\lambda(z;t,s,a) \tag{1.9}$$

$$|t|<|a|$$

When $t\to t/\lambda$ and $|\lambda|\to\infty$, (1.8) becomes

$$\sum_{n=0}^{\infty}\phi(z,s+n,a)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}\exp\left(\frac{t}{n+a}\right)=\psi(z,t,s,a),|t|<\infty \tag{1.10}$$

The extension of Hurwitz Lerch zeta function in multiparameter is defined by Srivastava [5] as follows.

$$\phi^{\left(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q\right)}_{\left(\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q\right)}(z,s,a)=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{n!\prod_{j=1}^{q}(\mu_j)_{n\sigma_j}}\bullet\frac{z^n}{(n+a)^s} \qquad (1.11)$$

$$p,q\in N_0;\lambda_j\in C\left(j=1,\ldots,p\right);\quad a,\mu_j\in c/z_0^-(j=1,\ldots,q)$$

$$\rho_j,\sigma_k\in R^+\left(j=1,\ldots,p,k=1,\ldots,q\right)$$

They also introduced the following generating relations associated with multiparameter Hurwitz- Lerch zeta function defined in (1.10)

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\phi^{(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}_{(\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q)}\left(z,s+n,a\right)t^n$$

$$=\sum_{k=0}^{\infty}\frac{E_k z^k}{(k+a)^{s-\lambda}(k+a-t)^{\lambda}}=\Omega_\lambda(z,t;s,a) \qquad |t|<|a| \qquad (1.12)$$

where

$$E_k=\frac{\prod_{j=1}^{p}(\lambda_j)_{k\rho_j}}{k!\prod_{j=1}^{q}(\mu_j)_{k\sigma_j}} \qquad (1.13)$$

$$k\in N_0,$$

The truncated form of the generating function $\Omega_\lambda\left(z,t;s,a\right)$ is also defined by Srivastava [6].

$$\Omega_\lambda^{0,r}(z,t;s,a)=\sum_{k=0}^{r}\frac{E_k z^k}{(k+a)^{s-\lambda}(k+a-t)^{\lambda}} \quad ,r\in N_0$$

$$\Omega_\lambda^{r+1,\infty}(z,t;s,a)=\sum_{r+1}^{\infty}\frac{E_k z^k}{(k+a)^{s-\lambda}(k+a-t)^{\lambda}} \quad ,r\in N_0$$

Which satisfy the following decomposition formula :

$$\Omega_\lambda^{(0,r)}\left(z,t;s,a\right)+\Omega_\lambda^{(r+1,\infty)}\left(z,t;s,a\right)=\Omega_\lambda\left(z,t;s,a\right) \qquad (1.14)$$

The integral representation formula for this generating functions is defined as follows (Srivasatava [5], [6]):

$$\Omega_\lambda\left(z,\omega;s,a\right)=\frac{1}{\Gamma s}\int_0^{\infty}t^{s-1}e^{-at}\,{}_p\psi_q^{*}\left[\begin{matrix}(\lambda_1,\rho_1),\ldots,(\lambda_p,\rho_p)\\(\mu_1,\sigma_1),\ldots,(\mu_q,\sigma_q)\end{matrix};ze^{-t}\right]{}_1F_1(\lambda;s;\omega t)dt$$

where $\left\{\min R(a),R(s)\right\}>0$ (1.15)

Results Required

The following results are required here [3, pp 181-184] :

For $f(t)=(\eta-\xi)+\rho(t-\xi)+\sigma(\eta-t)$ we have ,

$$\int_\xi^\eta\frac{(t-\xi)^{\upsilon}(\eta-t)^{\mu-1}}{\left[f(t)\right]^{\upsilon+\mu+1}}\,{}_2F_1\left[\begin{matrix}\zeta,b;\dfrac{(1+\sigma)(\eta-t)}{f(t)}\\\mu;\end{matrix}\right]dt$$

$$=\frac{(\eta-\xi)^{-1}(1+\rho)^{-\upsilon-1}(1+\sigma)^{\mu}\,\Gamma(\mu)\Gamma(\upsilon+1)\Gamma(\upsilon+\mu-\zeta-b+1)}{\Gamma(\upsilon+\mu-\zeta+1)\Gamma(\upsilon+\mu-b+1)}$$

$$\mathrm{Re}(\upsilon) > -1, \mathrm{Re}(\mu) > 0, \mathrm{Re}(\upsilon+\mu-b+1) > 0; \qquad (1.16)$$

$$\int_{\xi}^{\eta} \frac{(t-\xi)^{\mu}(\eta-t)^{\mu}}{[f(t)]^{2\mu+2}} {}_2F_1\left[\begin{matrix}\zeta, \; b; \\ \frac{1}{2}(\zeta+b+1);\end{matrix} \; \frac{(1+\sigma)(\eta-t)}{f(t)}\right]dt$$

$$= \frac{\pi\Gamma(\mu+1)\,\Gamma\left(\frac{\zeta+b+1}{2}\right)\,\Gamma\left(\mu+\frac{3-\zeta-b}{2}\right)}{2^{2\mu+1}(\eta-\xi)[(1+\sigma)(1+\rho)]^{\mu+1}\Gamma\left(\frac{(\xi+1)}{2}\right)\Gamma\left(\frac{(b+1)}{2}\right)} \times \frac{1}{\Gamma\left(\mu+\frac{3-\zeta}{2}\right)\Gamma\left(\mu+\frac{3-b}{2}\right)}$$

$$\mathrm{Re}(\mu) > -1, \mathrm{Re}(3-\zeta-b+2\mu) > 0; \qquad (1.17)$$

$$\int_{\xi}^{\eta} \frac{(t-\xi)^{\mu-\upsilon}(\eta-t)^{\mu-1}}{[f(t)]^{2\mu-\upsilon+1}} {}_2F_1\left[\begin{matrix}\zeta, 1-\xi; \\ \upsilon \;\; ;\end{matrix} \; \frac{(1+\sigma)(\eta-t)}{f(t)}\right]dt$$

$$= \frac{\pi\,\Gamma(\upsilon)\Gamma(\mu)\Gamma(\mu-\upsilon+1)}{2^{2\mu-1}(\eta-\xi)(1+\rho)^{1+\mu-\upsilon}(1+\sigma)^{\mu}\,\Gamma\left(\frac{1-\upsilon-\zeta}{2}\right)\Gamma\left(\frac{\upsilon+\zeta}{2}\right)} \times \frac{1}{\Gamma\left(\mu+\frac{\zeta-\upsilon+1}{2}\right)\Gamma\left(\mu+\frac{2-\zeta-\upsilon}{2}\right)}$$

$$\mathrm{Re}(\mu) > 0; \mathrm{Re}(\mu-\upsilon+1) > 0; \qquad (1.18)$$

We have established the following results involving functions related to Hurwtiz-Lerch Zeta functions.

2.Main Results

For $f(t)=(\eta-\xi)+\rho(t-\xi)+\sigma(\eta-t)$, $h(t)=(t-\xi)^{\gamma}[f(t)]^{-\gamma}$,

$v(t)=(t-\xi)^{\gamma}(\eta-t)^{\gamma}[f(t)]^{-2\gamma}$

and for the sequence of coefficient $\{E_k\}; k \in N_0$ the following main results are establish here

Result-1

$$\int_{\xi}^{\eta} \frac{(t-\xi)^{\upsilon}(\eta-t)^{\mu-1}}{[f(t)]^{\upsilon+\mu+1}} {}_2F_1\left[\begin{matrix}\zeta, b; \\ \mu;\end{matrix} \; \frac{(1+\sigma)(\eta-t)}{[f(t)]}\right] \Omega_{\lambda}(z, \omega h(t), s, a)\,dt$$

$$= \frac{(\eta-\xi)^{-1}(1+\rho)^{-\nu-1}(1+\sigma)^{\mu}\,\Gamma(\mu)\Gamma(\upsilon+1)\Gamma(\upsilon+\mu-\zeta-b+1)}{\Gamma(\upsilon+\mu-\zeta+1)\Gamma(\upsilon+\mu-b+1)}$$

$$\sum_{k=0}^{\infty} \frac{E_k z^k}{(a+k)^s} \; {}_3\psi_2^{*}\left[\begin{matrix}(\lambda,1),(\upsilon+1,\gamma),(\upsilon+\mu-\zeta-b+1,\gamma); \\ (\upsilon+\mu-\zeta+1,\gamma),(\upsilon+\mu-b+1,\gamma);\end{matrix} \; \frac{\omega}{(a+k)(1+\rho)^{\gamma}}\right] \qquad (2.1)$$

where $\eta \neq \xi$, $\min\{R(\upsilon), R(\mu)\} > 0; \gamma > 0$

Result-2

$$\int_{\xi}^{\eta}\frac{(t-\xi)^{\mu}(\eta-t)^{\mu}}{[f(t)]^{2\mu+2}}\,{}_2F_1\left[\begin{matrix}\zeta,b & ; & \dfrac{(1+\sigma)(\eta-t)}{f(t)}\\ \frac{1}{2}(\zeta+b+1); & & \end{matrix}\right]\Omega_{\lambda}\left(z,\omega v(t),s,a\right)dt$$

$$=\frac{\pi\Gamma(\mu+1)\Gamma\left(\frac{(\zeta+b+1)}{2}\right)\Gamma\left(\mu+\frac{3-\zeta-b}{2}\right)}{2^{2\mu}(\eta-\xi)(1+\sigma)^{\mu+1}(1+\rho)^{\mu+1}\Gamma\left(\frac{\zeta+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\mu+\frac{3-\zeta}{2}\right)\Gamma\left(\mu+\frac{3-b}{2}\right)}$$

$$\sum_{k=0}^{\infty}\frac{E_k z^k}{(a+k)^s}\,{}_2\psi_2^*\left[\begin{matrix}(\mu+1,\gamma),\left(\mu+\frac{3-\zeta-b}{2},\gamma\right) & ; & \dfrac{\omega}{4^{\gamma}(a+k)(1+\sigma)^{\gamma}(1+\rho)^{\gamma}}\\ \left(\mu+\frac{3-\zeta}{2},\gamma\right),\left(\mu+\frac{3-b}{2},\gamma\right); & & \end{matrix}\right] \qquad (2.2)$$

where $\eta\neq\xi; R(\mu)>0$

Result-3

$$\int_{\xi}^{\eta}\frac{(t-\xi)^{\mu-\upsilon}(\eta-t)^{\mu-1}}{[f(t)]^{2\mu-\upsilon+1}}\,{}_2F_1\left[\begin{matrix}\zeta,1-\zeta; & \dfrac{(1+\sigma)(\eta-t)}{f(t)}\\ \upsilon \quad ; & \end{matrix}\right]\Omega_{\lambda}\left(z,\omega v(t),s,a\right)dt$$

$$=\frac{\pi\Gamma\upsilon\Gamma\mu\Gamma(\mu+\upsilon+1)}{2^{2\mu-1}(\eta-\xi)(1+\rho)^{\mu-\upsilon+1}(1+\sigma)^{\mu}\Gamma\left(\frac{1-\upsilon-\zeta}{2}\right)\Gamma\left(\frac{\upsilon+\zeta}{2}\right)\Gamma\left(\mu+\frac{\zeta-\upsilon+1}{2}\right)\Gamma\left(\mu+\frac{2-\zeta-\upsilon}{2}\right)}$$

$$\sum_{k=0}^{\infty}\frac{E_k z^k}{(a+k)^s}\,{}_2\psi_2^*\left[\begin{matrix}(\mu+1,\gamma),\left(\mu+\frac{3-\zeta-b}{2},\gamma\right) & ; & \dfrac{\omega}{4^{\gamma}(a+k)(1+\sigma)^{\gamma}(1+\rho)^{\gamma}}\\ \left(\mu+\frac{3-\zeta}{2},\gamma\right),\left(\mu+\frac{3-b}{2},\gamma\right); & & \end{matrix}\right] \qquad (2.3)$$

where $\eta\neq\xi,\min[R\{\mu-\upsilon\},R(\mu)]>0;\gamma>0$

OUTLINES OF PROOFS

Proof of (2.1):

To prove the result in (2.1) first we denote its LHS by I_1, i.e.

$$I_1=\int_{\xi}^{\eta}\frac{(t-\xi)^{\upsilon}(\eta-t)^{\mu-1}}{[f(t)]^{\upsilon+\mu+1}}\,{}_2F_1\left[\begin{matrix}\zeta,\ b; & \dfrac{(1+\sigma)(\eta-t)}{f(t)}\\ \mu\ ; & \end{matrix}\right]\Omega_{\lambda}\left[z,\omega(t-\xi)^{\gamma}(\eta-t)^{\gamma}[f(t)]^{-2\gamma},s,a\right]dt$$

Now on using the definition of Ω_{λ} given in (1.12) and using the binomial expansion theorem and then on changing the order of integration and summation we have

$$I_1=\sum_{n,k=0}^{\infty}\frac{E_k z^k(\omega)^n(\lambda)_n}{(a+k)^{s+n}n!}\int_{\xi}^{\eta}\frac{(t-\xi)^{\upsilon+\gamma n}(\eta-t)^{\mu+n\gamma-1}}{[f(t)]^{\upsilon+\mu+1+2\gamma n}}\,{}_2F_1\left[\begin{matrix}\zeta, & b; & \dfrac{(1+\sigma)(\eta-t)}{f(t)}\\ & \mu; & \end{matrix}\right]dt$$

Now evaluating the inner integral with the help the of (1.16) and interpretate the *n*- series in view of (1.5) we at once arrive at the desired result in (2.1).

To prove the results of (2.2) and (2.3) we follow the similar lines as to prove the result (2.1) and using (1.17), (1.18) for results (2.2) and (2.3) respectively.

3. Particular cases:

1. If in results (2.1), (2.2) and (2.3), we take $p=1, \rho_1=1, \lambda=1$ and $q=0$ then we obtained the results involving $V_\lambda(z,t;s,a)$ defined in (1.8) .
2. If we assume in results (2.1), (2.2) and (2.3) then these results reduce to the known results due to Jaimini and Somani [4,pp. 6-7,eqs-(2.1)-(2.3)] respectively.

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