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Generalized Mittag-Leffler Function And Generalized Fractional Calculus Operators

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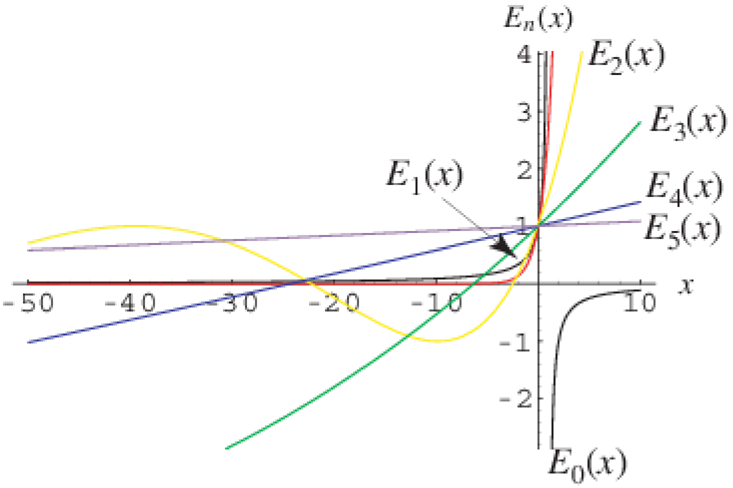
ABSTRACT: In complex analysis, Mittag-Leffler's theorem concerns the existence of meromorphic functions with prescribed poles. Conversely, it can be used to express any meromorphic function as a sum of partial fractions. It is sister to the Weierstrass factorization theorem, which asserts existence of holomorphic functions with prescribed zeros.

The theorem is named after the Swedish mathematician Gösta Mittag-Leffler who published versions of the theorem in 1876 and 1884.[1][2][3] Fractional calculus is a branch of mathematical analysis that studies the several different possibilities of defining real number powers or complex number powers of the differentiation operator

KEYWORDS: Mittag - Leffler, weierstrass factorization, holomorphic, Gosta –Mittag-Leffler, operator

I.INTRODUCTION

This paper deals with the study of a generalized function of Mittag-Leffler type. Various properties including usual differentiation and integration, Euler(Beta) transforms, Laplace transforms, Whittaker transforms, generalized hypergeometric series form with their several special cases are obtained and relationship with Wright hypergeometric function and Laguerre polynomials is also established.



The generalized fractional integrations of the generalized Mittag-Leffler type function (GMLTF) are established in this paper. The results derived in this paper generalize many results available in the literature and are capable of generating several applications in the theory of special functions. The solutions of a generalized fractional kinetic equation using the Sumudu transform is also derived and studied as an application of the GMLTF. Fractional calculus is one of the prominent branch of applied mathematics that deals with non-integer order derivatives and integrals (including complex orders), and their applications in almost all disciplines of science and engineering [18–22]. In this line, the use of special functions in connection with fractional calculus also studied widely [23–27]. For the basics of fractional calculus and its related literature, interesting readers can be referred to as Kiryakova [28], Miller and Ross [29], and Srivastava et al. [30]. The generalized fractional integrations of the generalized Mittag-Leffler type function is studied in this paper. The obtained results are expressed in terms of the generalized Wright hypergeometric function and generalized hypergeometric functions. To show the potential application of GMLTF, the solutions of fractional kinetic equations are derived with the help of Sumudu transform. The results obtained in this study have significant importance as the solution of the equations are general and can derive many new and known solutions of FKEs involving various type of special functions.

II.DISCUSSION

The paper is devoted to the study of the function $E^{\gamma}_{\rho,\mu}(z)$ defined for complex ρ, μ, γ (Re(ρ) > 0) by which is a generalization of the classical Mittag-Leffler function $E_{\rho,\mu}(z)$ and the Kummer confluent hypergeometric function Φ(γ, μ; z). The properties of $E^{\gamma}_{\rho,\mu}(z)$ including usual differentiation and integration, and fractional ones are proved. Further the integral operator with such a function kernel is studied in the space L(a, b). Compositions of the Riemann–Liouville fractional integration and differentiation operators with $E^{\gamma}_{\rho,\mu,\omega;a+}$ are established. An analogy of the semigroup property for the composition of two such operators with different indices is proved, and the results obtained are applied to construct the left inversion operator to the operator $E^{\gamma}_{\rho,\mu,\omega;a+}$. Since, for γ = 0, $E^{0}_{\rho,\mu,\omega;a+}$ coincides with the Riemann–Liouville fractional integral of order μ, the above operator and its inversion can be considered as generalized fractional calculus operators involving the generalized Mittag-Leffler function $E^{\gamma}_{\rho,\mu}(z)$ in the kernels. Similar assertions are presented for the integral operators containing the Mittag-Leffler and Kummer functions, $E_{\rho,\mu}(z)$ and Φ(γ, μ; z), in the kernels, and applications are given to obtain solutions in closed form of the integral equations of the first kind. The Mittag-Leffler function has gained importance and popularity during the last one and a half decades due to its direct involvement in the problems of physics, biology, engineering, and applied sciences. Mittag-Leffler function naturally occurs as the solution of fractional order differential equations and fractional order integral equations. A special role of the Mittag-Leffler functions in the fractional calculus has been discovered by many scientists from different view points. In 1899, Mittag-Leffler began the publication of a series of articles under the common title 'Sur la representation analytique d'une branche uniforme d'une fonction monogene' ('On the analytic representation of a single-valued branch of a monogene function') published mainly at 'Acta Mathematica'. His research was connected with the solution of a problem of analytic continuation of complex functions represented by power series. The function which he used for the solution of this problem was named later as the Mittag-Leffler function. Following the line of Mittag-Leffler's consideration, several investigations related to this function, its generalizations and related special functions have been done at the beginning of the 20th century (see, e.g., review article [10] and references therein). Probably for the first time, an interest to this function from the application appeared due to representation in terms of this function the solution of the Abel integral equation of the second order made by Hille and Tamarkin [12]. Nowadays this function and its numerous generalizations are used in different fractional models (see monographs listed above). A special role of the MittagLeffler function was pointed out by Kiryakova [23, 24], who included it into the class of special functions for fractional calculus. Moreover, based on the role of the Mittag-Leffler function in applications, Mainardi [25] called it 'the queen of fractional calculus'. Due to this exceptional role of the collection of Mittag-Leffler functions, any new exact result involving these functions seem very interesting. This paper is devoted to the properties of the so-called Marichev–Saigo–Maeda generalized fractional operator, i.e. integral transform of the Mellin convolution type with the Appell (or Horn) function F3. This operator was introduced nearly 40 years ago by Marichev [26] and studied in some recent papers, including the papers by Saigo and Maeda [39] and, Saigo and Saxena.

III.RESULTS

In applied mathematics and mathematical analysis, a fractional derivative is a derivative of any arbitrary order, real or complex. Its first appearance is in a letter written to Guillaume de l'Hôpital by Gottfried Wilhelm Leibniz in 1695.[2] Around the same time, Leibniz wrote to one of the Bernoulli brothers describing the similarity between the binomial theorem and the Leibniz rule for the fractional derivative of a product of two functions. Fractional calculus was introduced in one of Niels Henrik Abel's early papers[3] where all the elements can be found: the idea of fractional-order integration and differentiation, the mutually inverse relationship between them, the understanding that fractional-order differentiation and integration can be considered as the same generalized operation, and even the unified notation for differentiation and integration of arbitrary real order.[4] Independently, the foundations of the subject were laid by Liouville in a paper from 1832.[5][6][7] The autodidact Oliver Heaviside introduced the practical use of fractional differential operators in electrical transmission line analysis circa 1890.[8] The theory and applications of fractional calculus expanded greatly over the 19th and 20th centuries, and numerous contributors have given different definitions for fractional derivatives and integrals.[9] The classical form of fractional calculus is given by the Riemann–Liouville integral, which is essentially what has been described above. The theory of fractional integration for periodic functions (therefore including the "boundary condition" of repeating after a period) is given by the Weyl integral. It is defined on Fourier series, and requires the constant Fourier coefficient to vanish (thus, it applies to functions on the unit circle whose integrals evaluate to zero). The Riemann–Liouville integral exists in two forms, upper and lower. Unlike classical Newtonian derivatives, fractional derivatives can be defined in a variety of different ways that often do not all lead to the same result even for smooth functions. Some of these are defined via a fractional integral. Because of the incompatibility of definitions, it is frequently necessary to be explicit about which definition is used. In 2013–2014 Atangana et al. described some groundwater flow problems using the concept of a derivative with fractional order.[35][36] In these works, the classical Darcy law is generalized by regarding the water flow as a function of a non-integer order derivative of the piezometric head. This generalized law and the law of conservation of mass are then used to derive a new equation for groundwater flow. This equation has been shown useful for modeling contaminant flow in heterogenous porous media.[37][38][39]

Atangana and Kilicman extended the fractional advection dispersion equation to a variable order equation. In their work, the hydrodynamic dispersion equation was generalized using the concept of a variational order derivative. The modified equation was numerically solved via the Crank–Nicolson method. The stability and convergence in numerical simulations showed that the modified equation is more reliable in predicting the movement of pollution in deformable aquifers than equations with constant fractional and integer derivatives[40] In the fields of dynamical systems and control theory, a fractional-order system is a dynamical system that can be modeled by a fractional differential equation containing derivatives of non-integer order.[1] Such systems are said to have fractional dynamics. Derivatives and integrals of fractional orders are used to describe objects that can be characterized by power-law nonlocality,[2] power-law long-range dependence or fractal properties. Fractional-order systems are useful in studying the anomalous behavior of dynamical systems in physics, electrochemistry, biology, viscoelasticity and chaotic systems.[1] Exponential laws are a classical approach to study dynamics of population densities, but there are many systems where dynamics undergo faster or slower-than-exponential laws. In such case the anomalous changes in dynamics may be best described by Mittag-Leffler functions.[4]

Anomalous diffusion is one more dynamic system where fractional-order systems play significant role to describe the anomalous flow in the diffusion process.

Viscoelasticity is the property of material in which the material exhibits its nature between purely elastic and pure fluid. In case of real materials the relationship between stress and strain given by Hooke's law and Newton's law both have obvious disadvances. So G. W. Scott Blair introduced a new relationship between stress and strain

RESULTS

A fractional-order integrator or just simply fractional integrator is an integrator device that calculates the fractional-order integral or derivative (usually called a differintegral) of an input. Differentiation or integration is a real or complex parameter. The fractional integrator is useful in fractional-order control where the history of the system under control is important to the control system output. Fractional-order control (FOC) is a field of control theory that uses the fractional-order integrator as part of the control system design toolkit. The use of fractional calculus (FC) can improve and generalize well-established control methods and strategies. [1]

The fundamental advantage of FOC is that the fractional-order integrator weights history using a function that decays with a power-law tail. The effect is that the effects of all time are computed for each iteration of the control algorithm. This creates a 'distribution of time constants,' the upshot of which is there is no particular time constant, or resonance frequency, for the system.

Fractional-order control shows promise in many controlled environments that suffer from the classical problems of overshoot and resonance, as well as time diffuse applications such as thermal dissipation and chemical mixing. Fractional-order control has also been demonstrated to be capable of suppressing chaotic behaviors in mathematical models of, for example, muscular blood vessels.[3]

Initiated from the 80's by the Pr. Oustaloup's group, the CRONE approach is one of the most developed control-system design methodologies that uses fractional-order operator properties.

When l'Hˆopital1 asked what would be the result of half-differentiating a function, Leibnitz (1695)2 replied that "It leads to a paradox, from which one day useful consequences will be drawn." Heaviside's (1871)3 view was "There is universe of mathematics lying in between the complete differentiations and integrations, and that fractional operators push themselves forward sometimes, and are just as real as others." The process by which we arrive at fractional operators is somewhat like what was done for numbers. First we had positive integers, and then followed the zero, fractions, irrational, negative, and complex numbers. A scalar α raised to a fractional power such as 1/2 is understood in the context of the law of exponents, α nα m = α n+m, where n and m are numbers.

IV.CONCLUSIONS

Although α n, where n is a positive integer, is defined by α being multiplied by itself (n − 1) times, α 1/2 is defined by α 1/2α 1/2 = 1. √ 2 is merely a notation, but α 1/2 can be used with ease in algebraic manipulations, and can participate in binary operations such as addition, subtraction, multiplication, division, and exponentiation. Though the use of fractional operators and derivatives is wide-spread, by choice we will restrict ourselves to the areas of engineering related to mechanical systems. For convenience we will also assume that the numbers and functions treated here are real, although generalizations to complex numbers exist.

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